Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Determine maximum value of

$$F(x,y,z) = \min\left\{\frac{|y-z|}{|x|}, \frac{|z-x|}{|y|}, \frac{|x-y|}{|z|}\right\},$$

where x,y,z be arbitrary nonzero real numbers.

Solution1.(**Logical**)

First note that maximum value of $F(x,y,z)$ can be characterized as maximum value of parameter $t > 0$ for which inequality $t \le F(x,y,z)$ has solution in real nonzero x,y,z. Thus we should find greatest value of parameter t for which inequality

(1) $$t \le \min\left\{\frac{|y-z|}{|x|}, \frac{|z-x|}{|y|}, \frac{|x-y|}{|z|}\right\} \Leftrightarrow \left\{\begin{array}{l} |x|t \le |y-z| \\ |y|t \le |z-x| \\ |z|t \le |x-y| \end{array}\right|$$

has nonzero solution.

Since $F(x,y,z) = F(-x,-y,-z)$ Due full symmetry and Since $F(x,y,z) = F(-x,-y,-z)$ we can suppose that $x < y < z$ where $y,z > 0$. We will consider two cases:

1. In the case when $x < 0$ inequality (1) equivalent to the system of inequalities

(2) $$\left\{\begin{array}{l} -xt \le z-y \\ yt \le z-x \\ zt \le y-x \end{array}\right| .$$

Adding the first and the third inequalities of (2) we obtain

$t(z-x) \le z-x \Leftrightarrow t \le 1,$ since $z-x > 0$.

2. In the case $x > 0$ from inequality $|z|t \le |x-y| \Leftrightarrow zt \le y-x \Leftrightarrow z \le \dfrac{y-x}{t}$

and inequality $y < z$ follows $x < y(1-t) \Rightarrow t < 1$.

So, $t \le 1$ is the necessity condition for solvability of (1).

From the other hand, if we set $t = 1$ in the system (2) then for any positive p,q and $x =: -p,\ y := q$ we obtain for z inequalities $p \le z-q, q \le z+p, z \le q+p$ and $q < z$. But $p \le z-q \Leftrightarrow p+q \le z \Rightarrow q < z$ and $q \le z+p$. Thus for z remains only $p+q \le z \le q+p,$ i.e. $z = p+q$.

So, greatest value of parameter t which provide solvability of inequality (1) is 1.

Moreover if $t = 1$ then inequality (1) has infinitely many solutions $(-p,q,p+q)$ for arbitrary $p,q > 0$.

Thus we finally get $\max\limits_{x,y,z\neq 0}\left(\min\left\{\frac{|y-z|}{|x|}, \frac{|z-x|}{|y|}, \frac{|x-y|}{|z|}\right\}\right) = 1.$

Solution 2.(**Phenomenological**).

First we will prove that $F(x,y,z) \le 1$.

Really, if we suppose opposite, i.e. that there is $x,y,z \neq 0,$ such

$$\min\left\{\frac{|y-z|}{|x|},\frac{|z-x|}{|y|},\frac{|x-y|}{|z|}\right\} > 1 \iff \left\{\begin{array}{l} |y-z| > |x| \\ |z-x| > |y| \\ |x-y| > |z| \end{array}\right| \iff \left\{\begin{array}{l} (y-z)^2 - x^2 > 0 \\ (z-x)^2 - y^2 > 0 \\ (x-y)^2 - z^2 > 0 \end{array}\right|.$$

Multiplying all inequalities in the latter system we immediately obtain contradiction because $0 < \left((y-z)^2 - x^2\right)\left((z-x)^2 - y^2\right)\left((x-y)^2 - z^2\right) =$ $(y-z-x)(y-z+x)(z-x-y)(z-x+y)(x-y-z)(x-y+z) =$ $-(x+y-z)^2(y+z-x)^2(z+x-y)^2 < 0.$

Since $F(-1,2,3) = 1$ then upper bound 1 for $F(x,y,z)$ is its maximum.